

# M Theory Extensions of T Duality<sup>1</sup>

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## Abstract

T duality expresses the equivalence of a superstring theory compactified on a manifold  $K$  to another (possibly the same) superstring theory compactified on a dual manifold  $\tilde{K}$ . The volumes of  $K$  and  $\tilde{K}$  are inversely proportional. In this talk we consider two M theory generalizations of T duality: (i) M theory compactified on a torus is equivalent to type IIB superstring theory compactified on a circle and (ii) M theory compactified on a cylinder is equivalent to  $SO(32)$  superstring theory compactified on a circle. In both cases the size of the circle is proportional to the  $-3/4$  power of the area of the dual manifold.

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# 1 Introduction

It is a pleasure for me to speak on the occasion of the 60th birthday of my good friend, Keiji Kikkawa. I met Keiji during my first visit to the Aspen Center for Physics in the summer of 1969. This was shortly after the appearance of the famous paper of Kikkawa, Sakita, and Virasoro,[1] which introduced the idea of regarding the dual resonance model  $n$ -point functions as the tree approximation of a unitary quantum theory, rather than just as interesting functions for use in phenomenology. This seemed very novel and exciting to me at the time. In Aspen, Keiji and I collaborated in a study of the Regge asymptotic behavior of the KSV loop amplitudes.[2] I found the experience so pleasant that I have returned to Aspen almost every summer since then.

Many years later, after we had learned that the formulas of the earlier era described strings and were better suited to unification than to hadrons, Kikkawa and Yamasaki discovered the “T duality” arising from compactification of closed string theories on a circle.[3] This marked the beginning of a new line of attack on string theory that has proved to be extremely fruitful.[4] T dualities are now recognized as a particular class of discrete dualities that are valid order-by-order in perturbation theory (*i.e.*, they are perturbative), though T duality is non-perturbative on the world sheet. In my talk today I would like to describe two simple extensions of T duality in the context of “M theory.” (M theory is a conjectured quantum theory in eleven dimensions whose low-energy effective description is eleven-dimensional supergravity.) An interesting aspect of these extensions is that they incorporate some of the more modern non-perturbative duality symmetries.

The basic results that I wish to describe are represented pictorially in Fig. 1. This figure combines two well-known perturbative T dualities and two non-perturbative identifications. The perturbative T duality on the left side of the figure is between type IIB superstring theory on a circle and type IIA superstring theory on a circle of reciprocal radius.[5] The perturbative T duality on the right side of the figure relates the  $E_8 \times E_8$  heterotic string on a circle and the  $SO(32)$  heterotic string on a circle of reciprocal radius.[6] One non-perturbative fact is that the IIA theory in ten dimensions actually has a circular eleventh dimension whose radius is proportional to the  $2/3$  power of the string coupling constant.[7, 8] Another is that the  $E_8 \times E_8$  heterotic theory in ten dimensions actually has an eleventh dimension that is a line segment  $I$  (or, equivalently, a  $\mathbb{Z}_2$  orbifold of a circle).[9] The length of the line segment also scales as the  $2/3$  power of the string coupling constant.

What I propose to do in this lecture is to by-pass the IIA and  $E_8 \times E_8$  theories and discuss the following two dualities: 1) the equivalence of M theory on a two-torus  $T^2$  and IIB superstring theory on a circle  $S^1$ ; 2) the equivalence of M theory on a

cylinder  $C$  and  $SO(32)$  superstring theory on a circle  $S^1$ . In each case we will find that the area of the manifold  $T^2$  and  $C$  (at fixed shape) scales as the  $-4/3$  power of the size of the circle  $S^1$ . The wrapping of all relevant  $p$ -branes on the compact spaces will be considered. By requiring a detailed matching of  $p$ -branes in each pair of dual theories we achieve numerous consistency tests of the overall picture and deduce relations among various  $p$ -brane tensions. The discussion of M/IIB duality is a review of results reported previously,[10] whereas the M/ $SO(32)$  duality has not been presented previously.

## 2 BPS - Saturated p-Branes

A useful technique for obtaining non-perturbative information about superstring theories and M theory is to identify their BPS saturated  $p$ -branes.  $p$ -branes are  $p$ -dimensional objects that are characterized by a “tension”  $T_p$ , which is the mass per unit  $p$ -volume (and thus has dimensions of  $(mass)^{p+1}$ ), and a suitable  $(p+1)$ -dimensional world-volume theory. In theories with enough supersymmetry (all supersymmetric theories in ten or eleven dimensions, in particular) the tension of a  $p$ -brane carrying a suitable conserved charge has a strict lower bound proportional to that charge. When there is equality ( $T = Q$ ), the  $p$ -brane is said to be BPS saturated. Such  $p$ -branes belong to “short” representations of the supersymmetry algebra. So long as the supersymmetry is not broken, the tension of such a  $p$ -brane cannot be changed by any quantum correction – perturbative or non-perturbative. This generalizes the well-known fact that the photon belongs to a “short” representation of the Poincaré group and must remain massless so long as gauge invariance remains unbroken.

BPS saturated  $p$ -branes of superstring theories or M theory can be approximated by classical solutions of the corresponding effective supergravity theory that preserve half of the supersymmetry. Such solutions are not exact superstring solutions, of course, but they do demonstrate the existence of particular  $p$ -branes, give their tensions correctly, and exhibit other qualitatively correct properties. Generally these objects can be regarded as extremal black  $p$ -branes, *i.e.*, extremal black holes ( $p = 0$ ), extremal black strings ( $p = 1$ ), etc. In solving the supergravity equations to obtain  $p$ -brane configurations it is sometimes necessary to include a source to match a delta function singularity in the equations. When the source is required, one often speaks of a “fundamental  $p$ -brane” and when it isn’t of a “solitonic  $p$ -brane.” It will not be necessary for us to keep track of this distinction here, however.

The conserved charges carried by  $p$ -branes are associated with antisymmetric tensor gauge fields

$$A_n = A_{\mu_1 \dots \mu_n} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \dots \wedge dx^{\mu_n}. \quad (1)$$

When these undergo gauge transformations  $\delta A_n = d\Lambda_{n-1}$ , the field strength  $F_{n+1} = dA_n$  is invariant. A supergravity theory with such a gauge field typically has two kinds of BPS-saturated  $p$ -brane solutions. The electric  $p$ -brane has  $p = n - 1$  and the dual magnetic  $p$ -brane has  $p = D - n - 3$ . When it is possible to reformulate the supergravity theory in terms of a dual potential  $\tilde{A}$  (satisfying  $\tilde{F} = d\tilde{A} = *F$  plus possible interaction corrections) there is no essential distinction between “electric” and “magnetic”  $p$ -branes. The two are interchanged in the dual formulation. In the case of an electric  $p$ -brane, the gauge field couples to the  $p$ -brane world volume generalizing the well-known  $j \cdot A$  interaction of charged point particles. It is also worth noting that the charges of dual electric and magnetic  $p$ -branes satisfy a generalized Dirac quantization condition  $Q_E Q_M \in 2\pi\mathbb{Z}$ . [11]

Let us now consider specific examples, beginning with type IIB supergravity theory. This theory contains a complex scalar field  $\rho = \chi + ie^{-\phi}$ , and so the specification of a vacuum is characterized by the modulus

$$\rho_0 = \langle \rho \rangle = \frac{\theta_B}{2\pi} + \frac{i}{\lambda_B}, \quad (2)$$

where  $\lambda_B$  is the string coupling constant. (Note that the analogous formula for  $N = 4$  theories in four dimensions has  $\rho_0 = \theta/2\pi + i/\lambda^2$ .) The  $S$  duality group of type IIB superstring theory is an  $SL(2, \mathbb{Z})$  under which  $\rho$  transforms nonlinearly in the usual way. Therefore,  $\rho_0$  may be restricted to the usual fundamental region. The IIB theory also has a pair of two-form potentials  $B_{\mu\nu}^{(I)}$  that transform as a doublet of the  $SL(2, \mathbb{Z})$  duality group. Therefore, the associated electric one-branes (strings) carry a pair of  $B$  charges. Suitably normalized, they can be chosen to be a pair of relatively prime integers  $(q_1, q_2)$ . Setting  $\theta_B = 0$  (to keep things simple), the tensions (in the IIB string metric) are [10]

$$T_1^{(B)}(q_1, q_2) = \left( q_1^2 \lambda_B + \frac{q_2^2}{\lambda_B} \right)^{1/2} T_1^{(B)}. \quad (3)$$

In the canonical metric  $T_1^{(B)}$  is a constant, and thus  $T_1^{(B)} \sim \lambda_B^{-1/2}$  in the string metric. Note that in the string metric only  $T_1^{(B)}(1, 0)$  is finite as  $\lambda_B \rightarrow 0$ . Therefore it is natural to regard it as the fundamental string and the others as solitons, though they are all mapped into one another by the duality group. The dual five-branes carry a pair of magnetic  $B$  charges. The IIB theory also contains a four-form potential  $A_4$  with a self-dual field strength  $F_5$ . As a result, electric and magnetic charge are identified in this case and carried by a self-dual three-brane.

The M theory story is somewhat simpler, since the only antisymmetric tensor gauge field in eleven-dimensional supergravity is a three-form  $A_3$ . There are associated electric two-brane and magnetic five-brane solutions. It is apparently not possible to

replace  $A_3$  by a dual six-form potential, so the electric–magnetic distinction is meaningful in this case. (The two-brane is “fundamental” and the five-brane is “solitonic” in the sense described earlier.)

In the case of type I or heterotic strings the relevant low-energy theory is  $N = 1$   $D = 10$  supergravity coupled to an  $E_8 \times E_8$  or  $SO(32)$  super Yang–Mills multiplet. In this case the relevant antisymmetric tensor is a two-form  $B_{\mu\nu}$ . The associated electric one-brane is the heterotic string. There is also a five-brane, which is the magnetic dual of the heterotic string. Type I strings do not carry a conserved charge, they are not BPS saturated, and therefore they can break. For these reasons, they can only be described reliably at weak coupling when they are metastable. Now that we have described the relevant theories and their  $p$ -branes we can turn to the duality analysis.

### 3 M/IIB Duality

The duality described by the left side of Figure 1 relates M theory compactified on a two-torus and IIB superstring theory compactified on a circle. The torus is described by its area  $A_M$  (in the canonical eleven-dimensional metric) and a modular parameter  $\tau$ , which may be taken to lie in the fundamental region of the  $SL(2, \mathbb{Z})$  modular group. The corresponding parameters of the IIB theory are the modulus  $\rho_0$  and the circumference of the circle  $L_B$  (in the canonical ten-dimensional metric, which is the one that is invariant under  $SL(2, \mathbb{Z})$  transformations). To test the equivalence of these two constructions, we will examine the matching of all  $p$ -branes in nine dimensions. These include various wrappings of the ones identified in the preceding section as well as new  $p$ -branes that arise by Kaluza–Klein mechanisms. The possible  $p$ -brane wrappings are depicted in Figure 2. As the figure shows,  $p$ -branes, for any  $p$  between 0 and 5, can be obtained by suitable wrapping of a  $p'$ -brane in either M theory or IIB theory. The identifications are straightforward for  $p = 1, 2, 3, 4$ . They also work for  $p = 0, 5$ , but one must be careful to take account of Kaluza–Klein effects in these cases. (The Kaluza–Klein vector fields in nine dimensions – arising due to isometries – support additional electric zero-branes and magnetic five-branes.) When the matching is done correctly, one finds a one-to-one correspondence of  $p$ -branes and their tensions in nine dimensions. The details have been worked out previously.[10] Here, I will simply state the results and discuss their implications.

The most important result, perhaps, is that one must make the identification [10, 12]

$$\rho_0 = \tau. \tag{4}$$

Thus, the geometric  $SL(2, \mathbb{Z})$  modular group of the torus is identified with the non-perturbative  $S$ -duality group of the IIB theory! The canonical metrics  $g^{(M)}$  and  $g^{(B)}$

are related (after the compactifications) by

$$g^{(M)} = (A_M^{1/2} T_2^{(M)} / T_1^{(B)}) g^{(B)}. \quad (5)$$

Here,  $T_2^{(M)}$  is the M theory two-brane tension. Both  $T_2^{(M)}$  and  $T_1^{(B)}$  (introduced earlier) are constants that define scales and can be set to unity without loss of generality, though I will not do that. In terms of these constants, the compactification scales  $A_M$  and  $L_B$  are related by

$$(T_1^{(B)} L_B^2)^{-1} = \frac{1}{(2\pi)^2} T_2^{(M)} A_M^{3/2}. \quad (6)$$

Thus,  $L_B \sim A_M^{-3/4}$ . This means that if one compactifies the IIB theory on a circle and lets the size of the circle vanish, while holding  $\rho_0$  fixed, one ends up with M theory in eleven dimensions! Conversely, if one compactifies the M theory on a torus and lets the torus shrink to zero at fixed shape, one ends up with a chiral theory – IIB superstring theory – in ten dimensions.

The matching of  $p$ -branes and their tensions also yields a number of relations among the tensions. Not only does one learn the relation between M and IIB tensions in eq. (6), but also relations among the tensions of the M and IIB theories separately. For the M theory, one learns that the five-brane tension is proportional to the square of the two-brane tension

$$T_5^{(M)} = \frac{1}{2\pi} (T_2^{(M)})^2. \quad (7)$$

This formula implies that the product of electric and magnetic charges is the minimum value allowed by the quantization condition.[13] For the  $p$ -branes of the IIB theory one finds that all of their tensions can be expressed in terms of  $T_1^{(B)}$  and the moduli

$$T_1^{(B)}(q_1, q_2) = \Delta_q^{1/2} T_1^{(B)} \quad (8)$$

$$T_3^{(B)} = \frac{1}{2\pi} (T_1^{(B)})^2 \quad (9)$$

$$T_5^{(B)}(q_1, q_2) = \frac{1}{(2\pi)^2} \Delta_q^{1/2} (T_1^{(B)})^3, \quad (10)$$

where

$$\Delta_q = \left( q_1 - \frac{q_2 \theta_B}{2\pi} \right)^2 \lambda_B + \frac{q_2^2}{\lambda_B}. \quad (11)$$

Equations (8) and (11) generalize eq. (3) to include  $\theta_B \neq 0$ . Note that the tension of all RR  $p$ -branes  $\sim 1/\lambda_B$  in the string metric, as expected for D-branes.[15]

A number of amusing things are taking place in the  $p$ -brane matchings that gave these relations. Let me just mention one of them. In matching zero-branes there is a duality between the M theory two-brane and the IIB strings that works as follows:[10] The Kaluza–Klein excitations of the strings on a circle correspond to wrappings of the two-brane on the torus. Conversely, the wrappings of the  $SL(2, \mathbb{Z})$  family of strings on the circle correspond to the Kaluza–Klein excitations of the membrane on the torus.

Let us pause for a moment to consider the physical meaning of what we have shown. The claim is that M theory on  $T^2$  is the same thing as IIB theory on  $S^1$ . If one considers the common nine-dimensional theory, one might imagine asking the question “How many compact dimensions are there?” This question has two correct answers – one and two – depending on whether one thinks of M theory or IIB theory. This paradoxical situation has a simple resolution. Fields that describe “matter” in one picture can describe “metric” in the other and vice versa. This situation already occurs for more conventional T duality – say, for the heterotic string on a torus. In that case the duality mixes up internal components of the metric with those of the two-forms and  $U(1)$  gauge fields. What is new in the present case is that (i) the dual compact spaces have different dimensions, (ii) certain components of the IIB theory metric correspond to components of the three-form gauge field in M theory, (iii) certain components of the M theory metric correspond to the complex scalar  $\rho$  of the IIB theory. Despite these differences of detail, the basic concept is the same. Another important distinction, of course, is that the generalization of T duality considered here encodes non-perturbative features of the theory.

## 4 M/SO(32) Duality

Let us now consider the duality depicted in the right-hand portion of Figure 1. This is the equivalence of M theory compactified on a cylinder with  $SO(32)$  superstring theory compactified on a circle. Note that we do not specify whether the  $SO(32)$  theory is type I or heterotic. The reason, of course, is that they are different descriptions of a single theory,[8, 14] so it is both of them. The dilaton in one description is the negative of the dilaton in the other, and so the coupling constants are related by  $\lambda_H^{(O)} = (\lambda_I^{(O)})^{-1}$ , where  $O$  represents  $SO(32)$ ,  $H$  represents heterotic, and  $I$  represents type I.

Recall that the  $SO(32)$  theory has two BPS saturated  $p$ -branes: the heterotic string and its magnetic dual, which is a five-brane. As discussed earlier, the type I string is not BPS saturated and will not be considered in our analysis. The  $p$ -branes arising from compactification on a circle are straightforward to work out and are depicted in Figure 3. As in Section 3, one should also be careful to include  $p$ -branes

of Kaluza–Klein origin.

M theory, before compactification, has a two-brane and a five-brane, but new issues arise when one considers compactification on a manifold with boundaries, and so we must first get that straight. Consider first the Horava–Witten picture [9] – M theory on  $\mathbb{R}^{10} \times I$ , an eleven-dimensional space-time with two parallel ten-dimensional boundaries. This is the non-perturbative description of the  $E_8 \times E_8$  heterotic string, just as M theory on  $\mathbb{R}^{10} \times S^1$  describes the type IIA superstring. We know that  $E_8 \times E_8$  theory, regarded as ten-dimensional has a one-brane (the heterotic string) and a dual five-brane. So we must ask what these look like in eleven dimensions. There is only one sensible possibility. An  $E_8 \times E_8$  heterotic “string” is really a cylindrical two-brane with one boundary attached to each boundary of the space time. Thus, an  $E_8 \times E_8$  heterotic string can be viewed as a ribbon with one  $E_8$  gauge group living on each boundary. This seems to be the only allowed two-brane configuration, since any other would give rise to a two-brane that remains two-dimensional in the weak coupling limit in which the space-time boundaries approach one another. For the five-brane, the story is just the reverse. It must not terminate on the space-time boundaries, but can exist as a closed surface in the bulk. This is required so that it can give a five-brane and not a four-brane in the weak coupling limit. Subsequent reduction on a circle to nine dimensions gives the wrapping possibilities depicted in Figure 3.

The cylinder  $C$  has a height  $L_1$  and a circumference  $L_2$ . These are conveniently combined to give an area  $A_C = L_1 L_2$  and a shape ratio  $\sigma = L_1/L_2$ . The circumference of the circle for the  $SO(32)$  theory compactification is denoted  $L_O$  (in the heterotic string metric). Also, the  $p$ -brane tensions of the  $SO(32)$  theory are denoted  $T_1^{(O)}$  and  $T_5^{(O)}$ . Now, guided by Figure 3, we again match  $p$ -branes in nine dimensions, just as we did in the previous section. One finds that the analog of the identification  $\rho_0 = \tau$  is

$$\sigma = \lambda_H^{(O)} = (\lambda_I^{(O)})^{-1} = L_1/L_2. \quad (12)$$

This means that when the spatial cylinder is a thin ribbon ( $\sigma \ll 1$ ) the heterotic string is weakly coupled, whereas when it is a thin tube ( $\sigma \gg 1$ ), the type I string is weakly coupled. The analog of eq. (6) is

$$(T_1^{(O)} L_O^2)^{-1} = \frac{1}{(2\pi)^2} T_2^{(M)} A_C^{3/2} \sigma^{-1/2}. \quad (13)$$

Aside from the factor of  $\sigma^{-1/2}$  the equations look the same. As before, for fixed  $\sigma, L_O \sim A_C^{-3/4}$ . Thus, the  $SO(32)$  theory in ten dimensions can be obtained by shrinking the cylinder to a point.

As in Section 3, the  $p$ -brane matching in nine dimensions gives various tension



relations. The only new one is

$$T_5^{(O)} = \frac{1}{(2\pi)^2} \left( \frac{L_2}{L_1} \right)^2 (T_1^{(O)})^3. \quad (14)$$

Combining eqs. (12) and (14), one learns that in the heterotic string metric, where  $T_1^{(O)}$  is constant,  $T_5^{(O)} \sim (\lambda_H^{(O)})^{-2}$ , as expected for a soliton. On the other hand, in the type I string metric  $T_1^{(O)} \sim 1/\lambda_I^{(O)}$  and  $T_5^{(O)} \sim 1/\lambda_I^{(O)}$ , as expected for  $D$ -branes.[15]

In the case of M/IIB duality we found that the  $SL(2, \mathbb{Z})$  modular group of the torus corresponded to the  $S$ -duality group of the IIB theory. In the present case of M/ $SO(32)$  duality, the cylinder does not have an analogous modular group. However, the interchange  $L_1 \leftrightarrow L_2$  (or  $\lambda_I \leftrightarrow \lambda_H$ ) corresponds to the strong/weak duality transformation of the  $SO(32)$  theory that relates the perturbative heterotic limit to the perturbative type I limit.

## 5 Conclusion

On several occasions Keiji Kikkawa has pioneered concepts that have led to important advances in string theory. The lessons of T duality are still being learned. In particular, the generalization of T duality to M theory described here exhibits the non-perturbative equivalence of all known superstring theories, realizing a dream I have had for many years. Other generalizations of T duality have been found,[16] and there may still be more to come. We owe Keiji a debt of gratitude for pointing us in this direction.

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**D = 11**

**D = 10**

**D = 9**

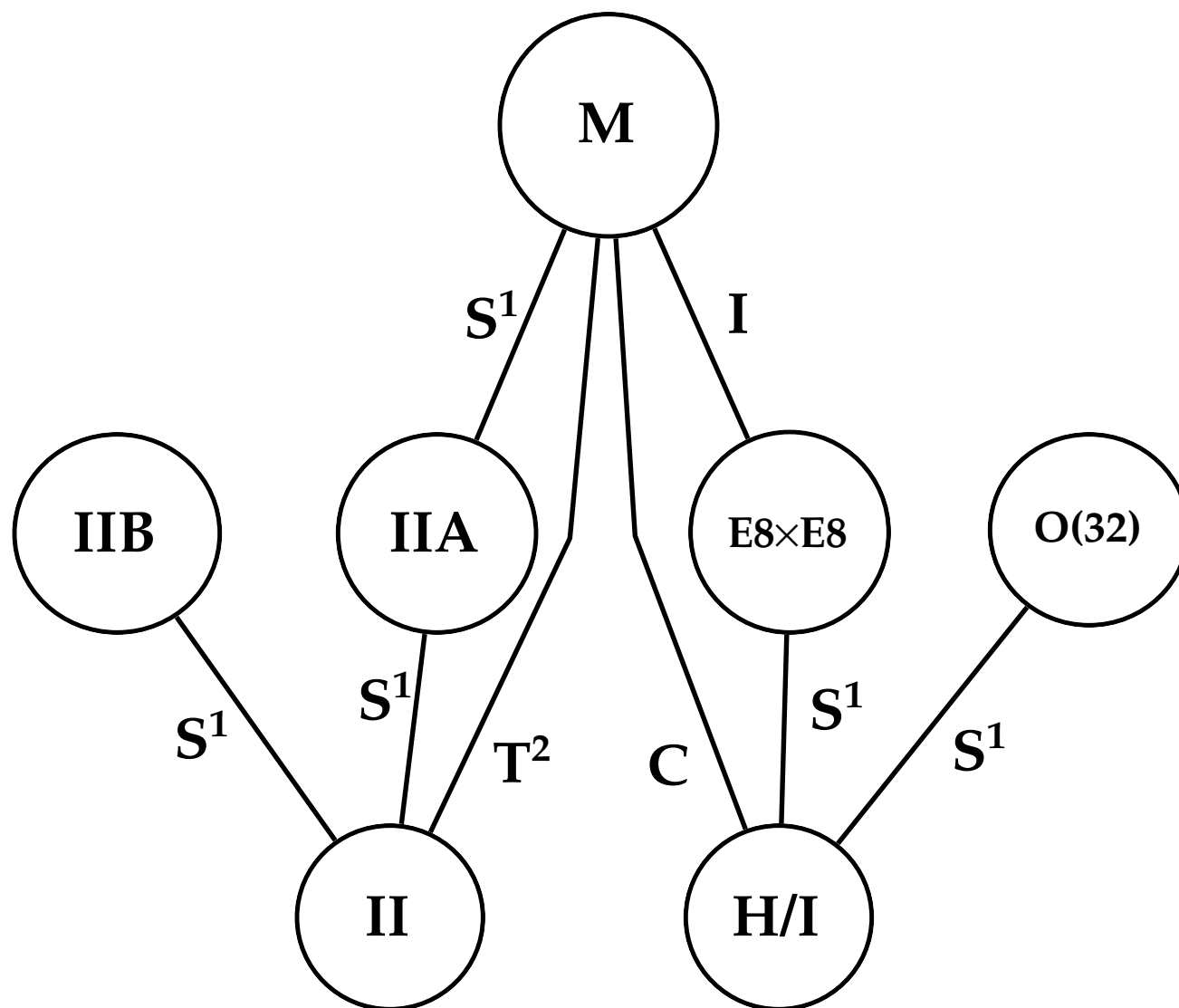


Figure 1

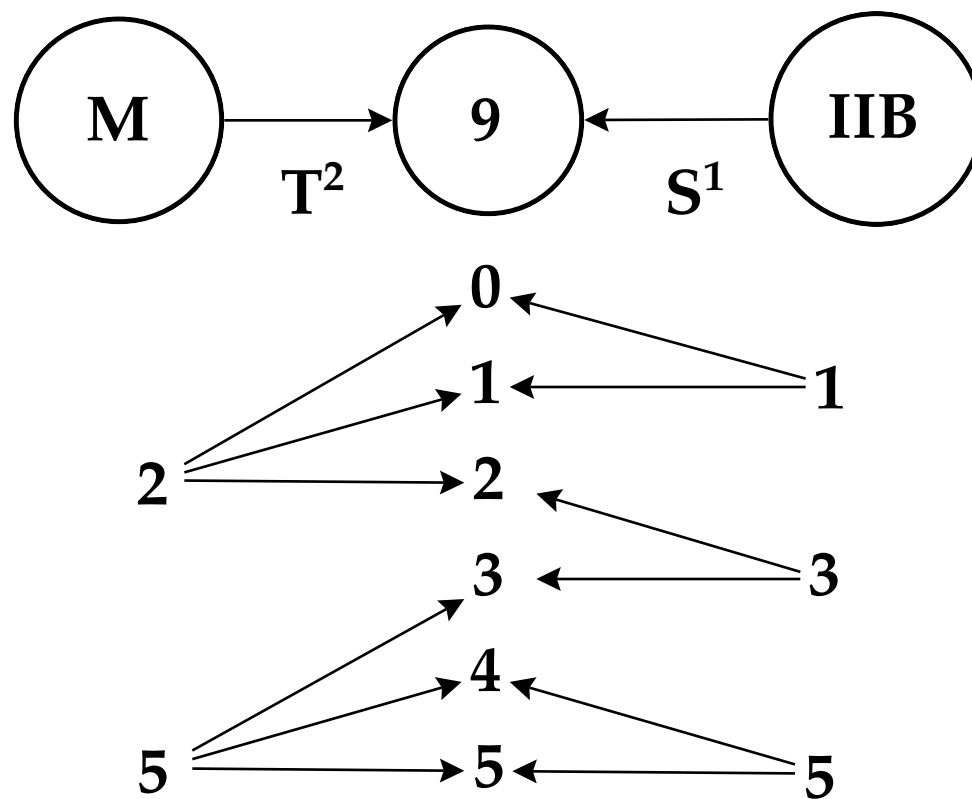


Figure 2

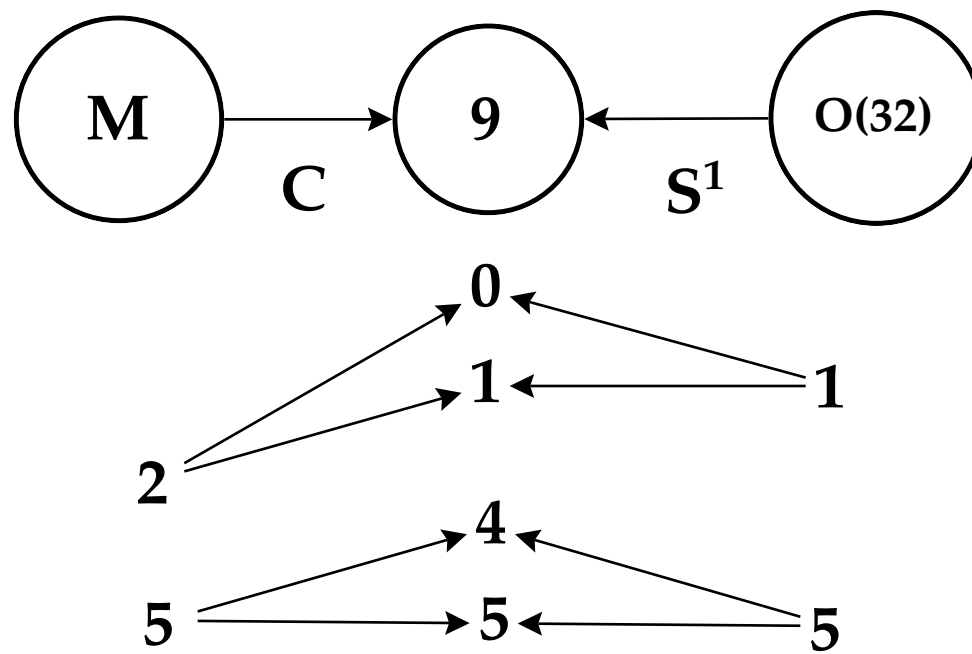


Figure 3